COMBINATORICA
Bolyai Society – Springer-Verlag

COMBINATORICA **24** (2) (2004) 209–232

PARAMETRIZED PARTITIONS OF PRODUCTS OF FINITE SETS*

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Received October 28, 2000

For every infinite sequence of positive integers $\{m_i\}_{i=0}^{\infty}$ and every Borel partition $c:\omega^\omega\times[\omega]^\omega\to\{0,1\}$ there is $H\in[\omega]^\omega$ and a sequence $\{H_i\}_{i=0}^{\infty}$ of subsets of ω, with $|H_i|=m_i$ for every i, such that c is constant on $(\prod_{i=0}^{\infty}H_i)\times[H]^\omega$.

1. Introduction

Infinite dimensional Ramsey Theory is an area of Ramsey Theory that deals with infinite *powers* of various structures. It has seen a considerable growth as well as a number of successful applications ever since its initiation in the early seventies with the work of Galvin–Prikry, Silver and Ellentuck (see, for example, [2,5,8,9,22–24]). The corresponding *polarized* theory of infinite *products* is a more recent venture to which we contribute with this paper (see [3,4,15,16,18] for previous work done in this direction). While the polarized theory on finite products of finite structures contains some of the finest results in the whole Ramsey Theory (see [9,11]), progress in the corresponding theory of either finite products of infinite structures or infinite products of finite or infinite structures has been much slower. Until quite recently, the theory only offered a few counterexamples pointing out more or less obvious restrictions. One such example, the mapping $c:\omega\times\omega\to 2$ defined by $c(m,n)=0$ if and only if $m<n$, or its infinite dimensional analogue

Mathematics Subject Classification (2000): 03E02, 03E15, 05D10

* Research partially supported by CNRS-FONACIT Project PI 2000001471.
† This author thanks the University of Paris VII for hospitality.

$c_\infty : \omega^\omega \to 2^\omega$, defined by $c_\infty(x)(n) = 0$ if and only if $x(n) < x(n+1)$, exhibit the maximally complex behavior of those functions on any product of the form $H_0 \times H_1$ or $H_0 \times H_1 \times \ldots$ respectively, of infinite subsets of ω. This example shows that the largest products $\prod_{i\in\omega} H_i$ on which a given mapping $f : \omega^\omega \to 2$ has a chance to be constant are those for which all except one of the H_i's are finite. That this is indeed true for all Borel mappings $f : \omega^\omega \to 2$ has been established by Henle ([12]). One way to view Henle's result is saying that the polarized partition relation

$$\begin{pmatrix} \omega \\ \omega \\ \vdots \end{pmatrix} \to \begin{pmatrix} m_0 \\ m_1 \\ \vdots \end{pmatrix} \tag{1}$$

in the class of Borel partitions (an immediate consequence of the Galvin–Prikry Theorem) can be "parametrized" by the 1-dimensional Ramsey Theorem $\omega \to (\omega)^1_2$. So it is natural to ask whether (1) can also be parametrized by higher-dimensional forms of Ramsey's Theorem, and in particular by the infinite-dimensional one, $\omega \to (\omega)^\omega$.

The purpose of this paper is to prove that the polarized partition relation (1) can indeed be parametrized by the the infinite-dimensional Ramsey partition relation $\omega \to (\omega)^\omega$, getting thus the best possible result in this direction. It turns out that the proof requires a refinement of (1) that is no longer a consequence of $\omega \to (\omega)^\omega$: For every infinite sequence of positive integers $\{m_i\}_{i=0}^\infty$ there is an infinite sequence of positive integers $\{n_i\}_{i=0}^\infty$ such that

$$\begin{pmatrix} n_0 \\ n_1 \\ \vdots \end{pmatrix} \to \begin{pmatrix} m_0 \\ m_1 \\ \vdots \end{pmatrix} \tag{2}$$

holds for Borel partitions. In [4] we give a bound on the n_i's in term of the m_i's. For example, an appropriate shift of the well-known Ackermann function as the sequence $\{n_i\}_{i=0}^\infty$ satisfies the partition relation (in the realm of Borel partitions) for a given primitive recursive sequence $\{m_i\}_{i=0}^\infty$. We show here that for every infinite sequence of integers $\{m_i\}_{i=0}^\infty$, there is a corresponding σ-field $\mathbb{PCL}(\{m_i\}_{i=0}^\infty)$ of subsets of $\omega^\omega \times [\omega]^\omega$ and an infinite sequence of positive integers $\{n_i\}_{i=0}^\infty$ with the property that for every $\mathbb{PCL}$-measurable mapping $f : (\prod_{i\in\omega} n_i) \times [\omega]^\omega \to \{0,1\}$ there exist a sequence $\{H_i\}_{i=0}^\infty$ with $H_i \subseteq n_i$ and $|H_i| = m_i$, and a set $H \in [\omega]^\omega$ such that f is constant on $(\prod_{i\in\omega} H_i) \times [H]^\omega$. We show that the σ-field $\mathbb{PCL}(\{m_i\}_{i=0}^\infty)$ is quite rich, and that, in particular, contains all Borel sets. In other words, the polarized partition relation (2), for a suitably chosen sequence $\{n_i\}$, admits parametrization by $\omega \to (\omega)^\omega$. This parametrized version of (2) is indeed a considerably stronger and much more difficult result. This can be seen, for example, noting that the fact that the field $\mathbb{PCL}(\vec{m})$ contains all F_σ subsets of $\omega^\omega \times [\omega]^\omega$ already implies the extension of the polarized partition relation (2) to all partitions determined by analytic subsets of ω^ω.

Some remarks seem pertinent in order to clarify the relationship of our results with some previous results in infinite-dimensional Ramsey Theory, for example, those contained in [5,6,8,17,20]. The optimal form of any of those previous results is obtained by first identifying the "basic sets", then defining the field of Ramsey sets as those subsets of the space that can be decided by taking "pure" extensions of basic sets, and finally proving that the field of Ramsey sets is in fact equal to the field of all subsets of the space that can be decided by taking extensions (not necessarily pure) of basic sets. Here, we have quite a different situation due to the fact that the partition relation (2) cannot be regarded as a "pigeon-hole principle" since the sequence $\{n_i\}$ is considerably faster than the sequence $\{m_i\}$. In other words, while we still have a natural notion of "basic subsets" of our space, the notion of Ramsey sets does not make sense in this context. Moreover, while the notion of pure extension between basic sets still makes sense in this new context, the corresponding notion of "rejection" (i.e. avoidance of the given subset of the space) is not monotone and this causes considerable difficulties not present in any of the situations considered previously. It should also be noted that our work hints to a quite general theory where the polarized version of any pigeon-hole principle (an analogue of (2)) can be parametrized by the infinite-dimensional version of any other pigeon-hole principle (an analogue of $\omega \to (\omega)^\omega$). In subsequent papers we plan to complete this line of research.

The paper is organized as follows. In section 2 we prove two combinatorial facts about partitions of finite products of finite sets which will be crucial for the rest of the paper. In particular, the two lemmas are used to set up a combinatorial machinery needed first to define the sequence $\{n_i\}_{i=0}^\infty$ in terms of the sequence $\{m_i\}_{i=0}^\infty$, and then to prove the parametrized version of the corresponding polarized partition relation (2). This is done in sections 3 and 4. In section 3, a family of infinite products of finite sets is defined along the lines of the definition of the Ackermann function, the main feature of which is the controlled rate of growth of the sizes of the finite factors. Section 4 is devoted to define a combinatorial forcing relation reminiscent of the Nash-Williams notions of "acceptance" and "rejection" used by Galvin and Prikry to prove the Borel version of $\omega \to (\omega)^\omega$ [8]. In section 5, for each infinite sequence of positive integers $\{m_i\}_{i=0}^{\infty}$, we define a σ-algebra $\mathbb{PCL}(\{m_i\}_{i=0}^{\infty})$ which is then used to prove the parametrized partition relation for Borel sets. Section 6 contains the proof of the main result.

Notation. We use conventional notation. The set of natural numbers is denoted by ω; $\omega^{<\omega}$ is used to denote the set of finite sequences of natural numbers, and ω^{ω} to denote the set of infinite sequences of natural numbers. For a set $A \subseteq \omega$, $[A]^{<\omega}$ is the set of its finite subsets, and $[A]^{\omega}$ is the set of its infinite subsets. If a, B are subsets of ω, $a \sqsubset B$ means that a is an initial segment of B. For $a \in [\omega]^{<\omega}$ and $A \in [\omega]^{\omega}$, $[a, A] = \{B \in [\omega]^{\omega} : a \sqsubset B \subseteq (a \cup A)\}$. Analogously, $[a, A]^{<\omega} = \{b \in [\omega]^{<\omega} : a \sqsubseteq b \subseteq (a \cup A)\}$. $A/b = \{n \in A : \forall k \in b (k < n)\}$ and $A/n = A/\{n\}$. The set ω^{ω} is given the product topology, considering ω as a discrete space. Given $s \in \omega^{\omega}$, $[s]$ denotes the basic neighborhood $\{x \in \omega^{\omega} : s \subset x\}$. The set $[\omega]^{\omega}$ is viewed as a subspace of ω^{ω}. An integer $n \in \omega$ will frequently be identified with the set $\{0, 1, \ldots, n-1\}$

We shall reserve the letters k, l, m, n for natural numbers, A, B, C for infinite subsets of ω, and a, b, c for elements of $[\omega]^{<\omega}$. We will also use $H_0, H_1, \ldots$ for finite subsets of ω. Sequences of natural numbers will be denoted with the letters s, t, r if they are finite, and with the letters x, y, z if infinite. For a finite sequence s, $|s|$ denotes the length of s, in other words, $|s| = n$ if $s = \{s(0), s(1), \ldots, s(n-1)\}$.

We will use $\vec{X}$ to denote the sequence of sets $\{X_i : i \in \omega\}$; in paritcular, sometimes we will write $\vec{m}$ instead of $\{m_i\}_{i=0}^{\infty}$ to denote a sequence of positive integers. Given $m_0, m_1, \ldots, n_0, n_1, \ldots$ and l, the partition symbol

$$\begin{pmatrix} n_0 \\ n_1 \\ \vdots \end{pmatrix} \to \begin{pmatrix} m_0 \\ m_1 \\ \vdots \end{pmatrix}_l$$

means that for every partition $f : \prod_{i \in \omega} n_i \to l$, there are sets $H_i \subseteq n_i$, $|H_i| = m_i$ for all i, such that f is constant on $\prod_{i \in \omega} H_i$. As customary, the subindex l of a partition symbol will be dropped when it equals 2.

If $\vec{H} = \{H_i : i \in \omega\}$ is a sequence of subsets of ω and $s \in \omega^{<\omega}$, we will use $[s, \vec{H}]$ to denote the set $\{x \in \omega^{\omega} : s = x \restriction |s| \text{ and } \forall i \geq |s| \ (x(i) \in H_i)\}$. We will use a similar notation even when dealing with finite sequences, so $[s, \vec{H}]^k = [s, \prod_{i<k} H_i]$ is the set of finite sequences $\{t \in \omega^k : s = t \restriction |s| \text{ and for } |s| \leq i < k, (x(i) \in H_i)\}$, and $[s, \vec{H}]^{<\omega} = \bigcup_{k \geq |s|} [s, \vec{H}]^k$.

2. Partitions of finite products of finite sets

In this section we prove two key combinatorial facts. The first one is from [4].

Lemma 2.1. *There is a function $g:\omega^{<\omega}\to\omega$ such that for every sequence $\{m_i : i\in\omega\}$ of natural numbers, for every $k\in\mathbb{N}$ and for every partition $c:\prod_{i\le k} g(m_0,\dots,m_i)\to 2$, there is a sequence of sets $\{H_i : i\le k\}$ such that $H_i\subseteq g(m_0,\dots,m_i)$, $|H_i|=m_i$, and c is constant on $\prod_{i\le k} H_i$.*

Proof. Given a sequence $\{m_i : i\in\omega\}$, the function g defined recursively by

$$g(m_0) = 2m_0 - 1, \quad \text{and}$$
$$g(m_0,\dots,m_{i+1}) = 2(m_{i+1}-1)\left[\prod_{k\le i}\binom{g(m_0,\dots,m_k)}{m_k}\right]+1,$$

satisfies the lemma. ∎

Recall that a subset $\mathcal{X}$ of $[\omega]^\omega$ is said to be Ramsey (or to have the Ramsey property) if there is an infinite $B\subseteq\omega$ such that $[B]^\omega$ is either included in $\mathcal{X}$ or is disjoint from $\mathcal{X}$. Recall also the result of Silver [22] saying that every analytic ($\boldsymbol{\Sigma}^1_1$) subset of $[\omega]^\omega$ is Ramsey. This result can be extended to the next level $\boldsymbol{\Sigma}^1_2$ of the projective hierarchy assuming either Martin's Axiom and the negation of the Continuum Hypothesis, or that there exist only countably many constructible reals (see [22]). We shall use the Ramsey property of $\boldsymbol{\Sigma}^1_2$ subsets of $[\omega]^\omega$ in the proof of the following lemma, which will be the crucial combinatorial tool in the next three sections. In section 6 we shall remove this assumption from the main result of this paper.

Lemma 2.2. *There is a function $h:\omega^{<\omega}\to\omega$ such that for every sequence $\{m_i\}_{i<\omega}$ of natural numbers and every partition*

$$c:\bigcup_{j<\omega}\prod_{i\le j} h(m_0,\dots,m_i)\to 2,$$

there is a sequence $\vec{H}=\{H_i\}_{i<\omega}$ with $H_i\subseteq h(m_0,\dots,m_i)$ and $|H_i|=m_i$ such that c is constant on $\prod_{i<j} H_i$ for infinitely many values of j.

Proof. It will be convenient to introduce some notation to express the combinatorial property that will be frequently used below. Given two sequences of natural numbers $\vec{m}=\{m_i\}_{i\in\omega}$ and $\vec{n}=\{n_i\}_{i\in\omega}$, the symbol

$$\vec{n}\to\vec{m}$$

denotes the statement:

"for every $c:\bigcup_{j<\omega}(\prod_{i<j} n_i)\to 2$, there is a sequence $\vec{H}=\{H_i\}_{i<\omega}$ with $H_i\subseteq n_i$ and $|H_i|=m_i$ such that c is constant on $\prod_{i<j} H_i$ for infinitely many values of j".

If $A\in[\omega]^\omega$, let A_{even} and A_{odd} be respectively the sequences $\{a_{2k}:k\in\omega\}$ and $\{a_{2k+1}:k\in\omega\}$, where $\{a_i:i\in\omega\}$ is the increasing enumeration of A.

Let $\mathcal{A}=\{A\in[\omega]^\omega : A_{odd}\to A_{even}\}$. Note that $\mathcal{A}$ is a $\boldsymbol{\Pi^1_2}$ subset of $[\omega]^\omega$. Applying the Ramsey property for $\boldsymbol{\Sigma^1_2}$ sets we find an infinite set $B\subseteq\omega$ such that $[B]^\omega\subseteq\mathcal{A}$ or $[B]^\omega\subseteq[\omega]^\omega\setminus\mathcal{A}$.

Case 1: $[B]^\omega\subseteq\mathcal{A}$. In this case, the mapping $h:\omega^{<\omega}\to\omega$ is defined as follows: if $\{b_i:i\in\omega\}$ is the increasing enumeration of B, put

$$h(m_0,\dots,m_j)=b_{2(\sum_{i=0}^{j} m_i)+1}.$$

Note that for every infinite sequence $\{m_i:i\in\omega\}$ of natural numbers, the set $A=\{b_{2(\sum_{i=0}^{j} m_i)}, b_{2(\sum_{i=0}^{j} m_i)+1}:j\in\omega\}$ is an infinite subset of B such that A_{even} pointwise dominates $\{m_i:i\in\omega\}$ and such that $\{h(m_0,\dots,m_j):j\in\omega\}$ coincides with $A_{odd}=\{b_{2(\sum_{i=0}^{j} m_i)+1}:j\in\omega\}$. So, by the fact that $A\in[B]^\omega\subseteq\mathcal{A}$, we have that the conclusion of the lemma is satisfied.

Case 2: $[B]^\omega\subseteq[\omega]^\omega\setminus\mathcal{A}$. We actually show that this case never happens, finishing this way the proof of the lemma.

Suppose such a set B exists, and consider the subset $\mathcal{C}$ of the product space $[B]^\omega\times 2^{(\omega^{<\omega})}$ consisting of all pairs (A,c) such that

- $A=\{a_i:i\in\omega\}$ is an infinite subset of B enumerated increasingly,
- $c:\omega^{<\omega}\to 2$,
- For every sequence $\{H_i:i\in\omega\}$ with $H_i\subseteq a_{2i+1}$ and $|H_i|=a_{2i}$ for all $i\in\omega$, the set $\{k\in\omega: c$ is constant on $\prod_{i=0}^{k} H_i\}$ is finite.

Note that $\mathcal{C}$ is a $\boldsymbol{\Pi^1_1}$ subset of the product space, so, by Kondo's uniformization theorem, we can find a mapping $A\mapsto c_A$ from $[B]^\omega$ into $2^{(\omega^{<\omega})}$ whose graph is a $\boldsymbol{\Pi^1_1}$ subset of $\mathcal{C}$.

Given $A\in[B]^\omega$ and its increasing enumeration $\{a_i:i\in\omega\}$, let $\hat{A}$ denote the subset of A given by $\{a_{\sigma(i)}\}$ where $\sigma:\omega\to\omega$ is the increasing mapping defined recursively as

$\sigma(0)=0,$

$\sigma(2k+2)=\sigma(2k+1)+1,$
$\sigma(2k+1)=1+2k+\sum_{j=0}^{k} 2a_{\sigma(2j)}.$

In other words, $\hat{A}$ takes the first element of A and if $m\in A$ is the element of A which occupies the $2k^{th}$ place in $\hat{A}$, between m and the next element of $\hat{A}$ there are $2m$ elements of A. Let the first m of these elements form a set called $H^0_k(A)$, and the next m elements form a set $H^1_k(A)$. Since neither of $\vec{H}^0(A)$ nor $\vec{H}^1(A)$ are homogeneous for $c_{\hat{A}}$, let

$l^0(A) = \max\{l : c_{\hat{A}}$ is constant on $\prod_{i=0}^{l} H^0_i(A)\}$, and
$l^1(A) = \max\{l : c_{\hat{A}}$ is constant on $\prod_{i=0}^{l} H^1_i(A)\}$.

Let $\mathcal{B} = \{A \in [B]^\omega : l^0(A) < l^1(A)\}$.

Applying the $\boldsymbol{\Sigma}^1_2$-Ramsey property, we find a set $X \in [B]^\omega$ such that $[X]^\omega \subseteq \mathcal{B}$ or $[X]^\omega \subseteq [\omega]^\omega \setminus \mathcal{B}$. Since the two cases are symmetric, it is enough to show that the first one is impossible.

Let $\{x_i : i \in \omega\}$ be the increasing enumeration of X, and let $Y = \{x_{\tau(i)} : i \in \omega\}$, where $\tau : \omega \to \omega$ is the function defined recursively as follows

$\tau(0) = 0$,
$\tau(2k+2) = \tau(2k+1) + 1$,
$\tau(2k+1) = 1 + 2k + x_{\tau(2k)} + f(x_{\tau(0)}, x_{\tau(2)}, \dots, x_{\tau(2k)})$,

where f is a function satisfying Lemma 2.1. For $i < \omega$, let H^1_i be the last $x_{\tau(2k)}$ elements of $X \cap (x_{\tau(2k)}, x_{\tau(2k+1)})$, and let N_i denote the rest of this set. Note that $|N_i| = f(x_{\tau(0)}, x_{\tau(2)}, \dots, x_{\tau(2k)})$. Since $Y \in [B]^\omega \subseteq \mathcal{A}^c$, there is a maximal integer l such that c_Y is constant on $\prod_{i \le l} H^1_i$. Consider the restriction of c_Y to $\prod_{i \le n+1} N_i$. By the property of the function f, there exist, for each $i \le l+1$, $H^0_i \subseteq N_i$, $|H^0_i| = x_{\tau(2i)}$, such that c_Y is constant on $\prod_{i \le l+1} H^0_i$. For $i > l+1$, let $H^0_i \subseteq N_i$, $|H^0_i| = x_{\tau(2i)}$ be chosen arbitrarily and let

$$A = Y \cup \Big(\bigcup_{i \in \omega} (H^0_i \cup H^1_i)\Big).$$

Then A is an infinite subset of X, $\hat{A} = Y$ and $H^\epsilon_i(A) = H^\epsilon_i$ for every $i \in \omega$ and $\epsilon < 2$. However, the maximal k for which $c_{\hat{A}} = c_Y$ is constant on $\prod_{i \le k} H^0_i(A) = \prod_{i \le k} H^0_i$ is at least equal to $l+1$. It follows that $A \notin \mathcal{B}$, which was to be shown. ∎

3. A family of Ackermannic products

Recall the Grzegorczyk sequence of primitive recursive functions given by $f_{n+1}(x) = f_n^{(x)}(1)$ (with $f_0(0) = 1, f_0(1) = 2$ and $f_0(x) = x+2$ for $x \ge 2$), and the definition of the corresponding diagonal function, the Ackermann function, given by $A(n) = f_n(n)$ (see [9]). Given a sequence $\vec{m}$ of natural numbers, using the function h of Lemma 2.2, we define a sequence of functions along the same lines.

Let h be a function satisfying Lemma 2.1 as well as Lemma 2.2 such that $h(m_0, \dots, m_i)$ is monotone in i and each of the arguments $m_0, \dots, m_i$.

Given a nondecreasing sequence $\vec{m} = \{m_i\}_{i=0}^{\infty}$, we define inductively a corresponding sequence of functions and their respective iterates.

Let $h_1(i) = h(m_0, \dots, m_i)$. We define the iterates of this function as follows. The first iterate is just the function h_1, $h_1^{[1]}(i) = h_1(i)$ for every i.

The second iterate is defined by

$$\begin{aligned} h_1^{[2]}(0) &= h(h_1(0)), \\ h_1^{[2]}(i) &= h(h_1(0), \dots, h_1(i)). \end{aligned}$$

In general, the $j+1$-th iterate is defined inductively by

$$\begin{aligned} h_1^{[j+1]}(0) &= h(h_1^{[j]}(0)), \\ h_1^{[j+1]}(i) &= h(h_1^{[j]}(0), \dots, h_1^{[j]}(i)). \end{aligned}$$

Suppose inductively that we have defined h_{k-1} as well as all of its iterates, proceed as follows to define h_k. Put $h_k(0) = h_1(0)$. If $h_k(0), h_k(1), \dots, h_k(i-1)$ have been defined, let $P(i,k) = |(\bigcup_{n<i} \prod_{j \le n} h_k(j)) \times (2^i + 1)|$, and let

$$h_k(i) = h_{k-1}^{[P(i,k)]}(i).$$

The iterates of the function h_k are defined as before,

$$\begin{aligned} h_k^{[1]} &= h_k, \\ h_k^{[j+1]}(0) &= h(h_k^{[j]}(0)), \\ h_k^{[j+1]}(i) &= h(h_k^{[j]}(0), \dots, h_k^{[j]}(i)). \end{aligned}$$

This completes the definition of the functions h_k.

Remark 3.1. The factor 2^i+1 can be removed from the definition of $P(i,k)$ as we shall need only the case of a single partition in Proposition 3.6 below. We include the more general definition just to point out the flexibility we actually have in defining the iterates of the basic map h.

The following fact about $h_k^{[j]}(n)$ will be used below frequently without further reference.

Lemma 3.2. *The sequence $h_k^{[j]}(n)$ is increasing in k, j, and n.*

Proof. a) First we show that for each k and j, $h_k^{[j]}(n)$ is increasing in n.

For $j = 1$, we have to show that for all n, $h_k(n+1) \ge h_k(n)$. This is verified by induction. For h_1, it follows by the definition of h. Assuming it is true for h_k, we have $h_{k+1}(0) = h_1(0)$ and $h_{k+1}(n+1) = h_k^{[P(n+1,k+1)]}(n+1)$, and one of the factors in the definition of $P(n+1,k+1)$ is $h_{k+1}(n)$, therefore $h_{k+1}(n+1)\geq h_{k+1}(n)$.

Assuming the inequality for j,

$$\begin{aligned} h_k^{[j+1]}(n+1) &= h(h_k^{[j]}(0),\ldots,h_k^{[j]}(n),h_k^{[j]}(n+1)) \\ &\geq h(h_k^{[j]}(0),\ldots,h_k^{[j]}(n)) \\ &= h_k^{[j+1]}(n). \end{aligned}$$

b) We show that $h_{k+1}^{[j]}(n)\geq h_k^{[j]}(n)$ by induction. For $j=1$ it follows from the definition of the functions h_k. And,

$$\begin{aligned} h_{k+1}^{[j+1]}(n) &= h(h_{k+1}^{[j]}(0),\ldots,h_{k+1}^{[j]}(n)) \\ &\geq h(h_k^{[j]}(0),\ldots,h_k^{[j]}(n)) \\ &= h_k^{[j+1]}(n). \end{aligned}$$

c) $h_k^{[j+1]}(n)=h(h_k^{[j]}(0),\ldots,h_k^{[j]}(n))\geq h_k^{[j]}(n)$. ∎

Using the functions h_k we define, for each nondecreasing sequence $\vec{m}$, a family $\mathcal{H}(\vec{m})$ of sequences of finite sets.

Definition 3.3. Let $\mathcal{T}$ be the collection of monotonically increasing functions $\xi:\omega\to\omega\setminus\{0\}$, such that $\xi(n+1)>\xi(n)$ for infinitely many n, and whenever this occurs, $\xi(n+1)=\xi(n)+1$. For $\xi\in\mathcal{T}$, h_ξ is the function defined by $h_\xi(i)=h_{\xi(i)}(i)$.

A sequence $\vec{H}=\{H_i:i\in\omega\}$ of finite subsets of ω is in $\mathcal{H}=\mathcal{H}(\vec{m})$ if there is a $\xi\in\mathcal{T}$ such that $|H_i|=h_\xi(i)$ for every $i\in\omega$. In this case, we say that $||\vec{H}||=\xi$. If $||\vec{H}||=\xi$, and $\xi(i-1)<\xi(i)$, we say that i is an expansion level of $\vec{H}$.

Recall that we use the letters $A,B,\ldots$ to denote elements of $[\omega]^\omega$, $a,b,\ldots$ will denote elements of $[\omega]^{<\omega}$, and $s,t,\ldots$ will be used for elements of $\omega^{<\omega}$.

Definition 3.4. Given $\vec{H},\vec{H}'\in\mathcal{H}$, $\vec{H}'\leq\vec{H}$ means that for every $i\in\omega$, $H_i'\subseteq H_i$, and $\vec{H}'\leq_n\vec{H}$ means that $\vec{H}'\leq\vec{H}$ and $H_i'=H_i$ for $i<n$.

Definition 3.5. Given $\vec{H},\vec{H}'\in\mathcal{H}$ and $A,A'\in[\omega]^\omega$, $(\vec{H}',A')\leq(\vec{H},A)$ if $\vec{H}'\leq\vec{H}$ and $A'\in[A]^\omega$. For $n\in\omega$, $(\vec{H}',A')\leq_n(\vec{H},A)$ is used when $(\vec{H}',A')\leq(\vec{H},A)$, $H_i'=H_i$ for $i<n$ and for each $i<n$, the ith element of A' equals the ith element of A (relative to the enumeration of these two sets).

The classes $\mathcal{H}(\vec{m})$ were defined in order to allow iterations in the use of Lemma 2.2. This is explained by the following fact.

Proposition 3.6. *Suppose $\vec{H}\in\mathcal{H}(\vec{m})$ and $A\in[\omega]^\omega$. Let n be an expansion level of $\vec{H}$ and suppose that for every $a\subseteq\{0,1,\dots,n-1\}$ we are given a partition*

$$c_a:\bigcup_{k<\omega}\prod_{i<k}H_i\to 2.$$

Then, there exists $\vec{J}\leq_n\vec{H}$ and $B\in[A]^\omega$ such that

$$c_a\restriction\bigcup_{k\in B}[s,\vec{J}]^k$$

is constant for every $s\in\prod_{i<n}H_i$ and every $a\subseteq\{0,1,\dots,n-1\}$.

Proof. Let $\xi\in\mathcal{T}$ be such that $||\vec{H}||=\xi$. Let $(s_1,a_1),\dots,(s_d,a_d)$ be a list of all pairs (s,a) where $s\in\prod_{i<n}H_i$ and $a\subseteq n$. Then, by our choice of the numbers $P(i,k)$,

$$|H_p|=h_\xi(p)\geq h^{[d]}_{\xi(p)-1}(p)$$

for all $p\geq n$. Define

$$c^1:\bigcup_{k\geq n}[s_1,\vec{H}]^k\to 2$$

by letting $c^1(t)=c_{a_1}(t\restriction l)$ where l is the maximal member of A smaller or equal to the length of t. By Lemma 2.2, there exists an infinte set $A^1\subseteq A/n$, and $\vec{H}^1\leq_n\vec{H}$ such that

$$|H^1_p|\geq h^{[d-1]}_{\xi(p)-1}(p)\text{ for all }p\geq n,$$

and such that

$$c^1\restriction\bigcup_{k\in A^1}[s_1,\vec{H}^1]^k\text{is constant.}$$

Define

$$c^2:\bigcup_{k\geq n}[s_2,\vec{H}^1]^k\to 2$$

by letting $c^2(t)=c_{a_2}(t\restriction l)$ where l is the maximal element of A^1 smaller or equal to the length of t. By Lemma 2.2 there is an infinite $A^2\subseteq A^1$ and $\vec{H}^2\leq_n\vec{H}^1$ suth that

$$|H^2_p|\geq h^{[d-2]}_{\xi(p)-1}(p)$$

for all $p\geq n$ and such that

$$c^2\restriction\bigcup_{k\in A^2}[s_2,\vec{H}^2]^k$$

is constant, and so on. After d steps of this procedure, we arrive at $\vec{H}^d \le_n \vec{H}^{d-1} \le_n \cdots \le_n \vec{H}$ and infinite $A^d \subseteq A^{d-1} \subseteq \cdots \subseteq A$ such that $|H^d_p| = h_{\xi(p)-1}(p)$ for all $p \ge n$ and such that

$$c_a \restriction \bigcup_{k\in A^d} [s, \vec{H}^d]^k$$

is constant for all $a \subseteq \{0,1,\dots,n-1\}$ and $s \in \prod_{i<n} H_i$. Then $\vec{J} = \vec{H}^d$ and $B = A^d$ satisfy the conclusion of the Proposition. ∎

Definition 3.7. A sequence $\{\vec{H}^j\}_{i=0}^{\infty}$ is called a fusion sequence if for some sequence $\{n_i\}_{i=0}^{\infty}$ of positive integers which is increasing and not eventually constant,

$$\vec{H}^0 \ge_{n_0} \vec{H}^1 \ge_{n_1} \dots \ge_{n_{j-1}} \vec{H}^j \ge_{n_j} \dots$$

and for infinitely many i's there is an expansion level j of $\vec{H}^{i+1}$ such that $n_i < j \le n_{i+1}$.

In this case, for every i, the sequence of finite sets $\{H^j_i\}_{j=0}^{\infty}$ is eventually constant; call this constant value H_i. Then $\vec{H} = \{H_i\}_{i=0}^{\infty}$ is in $\mathcal{H}$ and it is called the limit of the fusion sequence $\{\vec{H}^k\}_{k\in\omega}$.

Proposition 3.8. *For every $l \in \omega$, let $O_l \subseteq \omega^\omega$ be an open set. Given $\vec{H} \in \mathcal{H}(\vec{m})$ and $n \in \omega$, there is $\vec{J} \le_n \vec{H}$ such that for every $l \in \omega$, $O_l \cap \prod_{i\in\omega} J_i$ is clopen in $\prod_{i\in\omega} J_i$.*

Proof. First we prove that for an open set $\mathcal{O} \subseteq \omega^\omega$, there is $\vec{J} \le_n \vec{H}$ such that $\mathcal{O} \cap \prod_{i<\omega} J_i$ is clopen.

Let $\mathcal{O} \subseteq \omega^\omega$ be open, and let $\bar{n}$ be the first expansion level of $\vec{H}$ above n. Defina a partition

$c : \bigcup_{j\ge\bar{n}} \prod_{i<j} H_i \to 2$ by

$$c(t) = \begin{cases} 0 & \text{if } [t, \vec{H}] \not\subseteq \mathcal{O}; \\ 1 & \text{if } [t, \vec{H}] \subseteq \mathcal{O}. \end{cases}$$

By Proposition 3.6, there is $\vec{J} \le_{\bar{n}} \vec{H}$ such that for every $s \in \prod_{i<\bar{n}} H_i$, c is constant on $\bigcup_{j\in A} [s, \prod_{i<j} J_i] = [s, \vec{J}]^j$ for some infinite set A.

If for a given $s \in \prod_{i<\bar{n}} H_i$, the constant value of c is 1, then $[s, \vec{J}] \subseteq \mathcal{O}$; otherwise, $[s, \vec{J}] \cap \mathcal{O} = \emptyset$ (since the complement of $\mathcal{O}$ is closed).

Now, let $\{\mathcal{O}_l\}_{l=0}^{\infty}$ be a sequence of open subsets of ω^ω. Using what we have just shown, we build inductively a sequence $n_1 < n_2 < \dots$ of positive integers and a fusion sequence $\vec{H} \ge_{n_0} \vec{H}^0 \ge_{n_1} \vec{H}^1 \ge_{n_2} \dots \ge_{n_k} \vec{H}^k \ge_{n_{k+1}} \dots,$

such that for every $l\in\omega$, for every $s\in\prod_{i<n_l}$, $[s,\vec{H}^l]\subseteq\mathcal{O}_l$ or $[s,\vec{H}^l]\cap\mathcal{O}_l=\emptyset$, and therefore, $\vec{H}^l\cap\mathcal{O}_l$ is clopen. If $\vec{J}$ is the limit of the fusion sequence, then $\mathcal{O}_l\cap\prod_{i\in\omega}J_i$ is clopen for every l. ∎

4. Combinatorial forcing

In this section O will be a fixed subset of the space $\omega^\omega\times[\omega]^\omega$, and $\vec{m}=\{m_i\}_{i=0}^\infty$ a fixed monotone sequence of positive integers, and we are working under the notation set up in section 3 and under the assumption of Lemma 2.2. Inspired by the Nash-Williams proof of his famous generalization of Ramsey's theorem ([19]), we make the following definition.

Definition 4.1. Given $\vec{H}\in\mathcal{H}=\mathcal{H}(\vec{m})$, $A\in[\omega]^\omega$, $s\in\omega^{<\omega}$ and $a\in[\omega]^{<\omega}$, we say that $(\vec{H},A)$ accepts (s,a) if $[s,\vec{H}]\times[a,A]\subseteq O$. We say that $(\vec{H},A)$ rejects (s,a) if no pair $(\vec{H}',A')$ with $(\vec{H}',A')\leq_{|s|}(\vec{H},A)$ accepts (s,a). We say that $(\vec{H},A)$ decides (s,a) if it either accepts or rejects (s,a).

Lemma 4.2. *1) If $(\vec{H},A)$ accepts (rejects) (s,a) then the same holds for any pair $(\vec{H}',A')\leq_{|s|}(\vec{H},A)$. 2) If for some $n\in\omega$, $(\vec{H},A)$ accepts (s,a) for every $s\in\prod_{i<n}H_i$, then $(\vec{H},A)$ accepts $(\emptyset,a)$. 3) For every pair $(\vec{H},A)$ and for every (s,a), there is a pair $(\vec{H}',A')\leq_{|s|}(\vec{H},A)$ which decides (s,a).*

Proof. The three clauses follow immediately from the definition. ∎

Lemma 4.3. *For every $(\vec{H},A)$, for every $a\in[\omega]^{<\omega}$ and for every m, there is $(\vec{H}',A')\leq_m(\vec{H},A)$ such that $(\vec{H}',A')$ decides (s,a) for every $s\in\prod_{i<m}H'_i$.*

Proof. List as $s_1,s_2,\ldots,s_d$ the elements of the product $\prod_{i<p}H_i$.

Use part 3) of the previous lemma d times to get $(\vec{H},A)\geq_p(\vec{H}^1,A_1)\geq_p(\vec{H}^2,A_2)\geq_p\cdots\geq_p(\vec{H}^d,A_d)$ such that for every $j\leq d$, $(\vec{H}^j,A_j)$ decides (s_j,a). Define $\vec{H}'$ by $H'_i=H_i$ for $i<p$, and $H'_i=H^d_i$ for $i\geq p$; put $A'=A_d$. The pair $(\vec{H}',A')$ satisfies the desired properties. Note that p will not necessarily be an expansion level of $\vec{H}'$, nevertheless, $\vec{H}'\in\mathcal{H}$. ∎

Lemma 4.4. *Suppose $\vec{H}\in\mathcal{H}(\vec{m})$ and $A\in[\omega]^\omega$. Then, for every integer n there exists $(\vec{J},B)\leq_n(\vec{H},A)$, and an infinite set $M\subseteq\omega$ such that for every $k\in M$, $s\in\prod_{i<k}J_i$ and $b\subseteq B$, with $\max(b)<k$, $(\vec{J},B)$ decides (s,b).*

Proof. Let p_0 be the $(n+1)$-st element of A, and let $a=A\cap\{0,1,\ldots,p_0-1\}$. Pick an integer n_0 above the first expansion level of $\vec{H}$ which comes after

n. By Lemma 4.3, find $\vec{H}' \leq_{n_0} \vec{H}$ and $A_1 \in [A]^\omega$ such that $(\vec{H}^1, A_1)$ decides (s,b) for all $s \in \prod_{i<n_0} H_i$ and $b \subseteq a$. Let $p_1 = \min(A_1)$, and let n_1 be an integer above the first expansion level of $\vec{H}^1$ which comes after n_0, and so on. Suppose we have defined $\vec{H} \geq_{n_0} \vec{H}^1 \geq_{n_1} \cdots \geq_{n_k} \vec{H}^k$, and $A \supseteq A_1 \supseteq \cdots \supseteq A^k$, such that if $p_i = \min(A_i)$ $(i \leq k)$, then $p_0 < p_1 < \cdots < p_k$.

Pick n_{k+1} above the first expansion level of $\vec{H}^k$ above n_k. By Lemma 4.3, there exists $\vec{H}^{k+1} \leq_{n_{k+1}} \vec{H}^k$ and $A_{k+1} \in [A_k]^\omega$ such that $(\vec{H}^{k+1}, A_{k+1})$ decides (s,b) for all $s \in \prod_{i<n_{k+1}} H_i^k$ and $b \subseteq a \cup \{p_0, \dots p_k\}$. Let $\vec{J}$ be the limit of the fusion sequence $\{\vec{H}^k\}$, let $M = \{n_k : k \in \omega\}$ and let $B = \{p_k : k \in \omega\}$. Then $\vec{J}$, B and M satisfy the conclusion of the lemma. ∎

The following fact will actually not be needed in the proof of the main result of this paper. It is included here in order to point out the amount of control one has on the notion of rejection in this context which, unlike the analogous notion of rejection of Nash-Williams [19] and of Galvin–Prikry [8], lacks monotonicity, which is one of the main difficulties behind the proof of Lemma 5.8 below.

Lemma 4.5. *Suppose $\vec{H} \in \mathcal{H}(\vec{m})$ and $A \in [\omega]^\omega$ are such that for some infinite set M, the pair $(\vec{H}, A)$ decides (s,a) for $s \in \bigcup_{k \in M} \prod_{i<k} H_i$ and $a \subseteq A$ with $\max(a) < |s|$. Let $n \in M$ and let $\bar{n}$ be the first expansion level of $\vec{H}$ above n. Then there exists $\vec{J} \leq_{\bar{n}} \vec{H}$ and an infinite set $L \subseteq M \setminus \bar{n}$ such that for every $\bar{s} \in \prod_{i<\bar{n}} H_i$, $a \subseteq A \restriction \bar{n}$, either $(\vec{H}, A)$ accepts (t,a) for all $t \in \bigcup_{k \in L} [\bar{s}, \vec{J}]^k$, or $(\vec{H}, A)$ rejecs (t,a) for all $t \in \bigcup_{k \in L} [\bar{s}, \vec{J}]^k$. Moreover, if for some $s \in \prod_{i<k} H_i$ and $a \subseteq A \restriction n$, the pair $(\vec{H}, A)$ rejects (s,a), then there must be $\bar{s} \in \prod_{i<\bar{n}} H_i$ extending s such that $(\vec{H}, A)$ rejects (t,a) for all $t \in \bigcup_{k \in L} [\bar{s}, \vec{J}]^k$*

Proof. For $a \subseteq A \restriction \bar{n}$, define

$$c_a : \bigcup_{k \geq \bar{n}} \prod_{i<k} H_i \to 2$$

by letting $c_a(t) = 0$ if and only if the maximal element k of M that is $\leq |t|$ is bigger than the maximum of a and $(t \restriction k, a)$ is accpeted by $(\vec{H}, A)$. Applying Proposition 3.6 we get $\vec{J} \leq_{\bar{n}} \vec{H}$ and infinite $L \subseteq K \setminus \bar{n}$ satisfying the first part of the conclusion of the Lemma.

To see the second part of the conclusion, suppose that for some $s \in \prod_{i<n} H_i$ and $a \subseteq A \restriction n$ we have that for all $\bar{s} \in \prod_{i<\bar{n}} H_i$ extending s, the pair

$(\vec{H},A)$ accepts (t,a) for all $t\in\bigcup_{k\in L}[s',\vec{J}]^k$. It follows in particular that

$$[s,\vec{H}]\times[a,A]=\bigcup_{\bar{s}\in[s,\vec{H}]^{\bar{n}}}[\bar{s},\vec{H}]\times[a,A]\subseteq\mathcal{O}.$$

So, $(\vec{H},A)$ could not have rejected (s,a). ∎

5. A field of subsets of the product space

In this section, $\vec{m}$ will be a fixed sequence of positive integers. We will work under the assumption of Lemma 2.2, and use the notation of section 3. We will define a field $\mathbb{PCL}=\mathbb{PCL}(\vec{m})$ of subsets of $\omega^\omega\times[\omega]^\omega$ and show that it contains the closed sets and that it is closed under countable unions. From this it will be easy to deduce an optimal parametrized version of the Galvin–Prikry theorem.

Definition 5.1. Let $\mathbb{PCL}=\mathbb{PCL}(\vec{m})$ be the collection of subsets O of the product space $\omega^\omega\times[\omega]^\omega$ for which the following holds: For every $\vec{H}\in\mathcal{H}=\mathcal{H}(\vec{m})$, $A\in[\omega]^\omega$, $n\in\omega$ and $a\in[A]^{<\omega}$, there are $\vec{H}'\leq_n\vec{H}$, $A'\in[a,A]$, and an integer l above n such that for every $s\in\prod_{i<l}H'_i$,

$$[s,\vec{H}']\times[a,A']\subseteq O \quad\text{or}$$

$$[s,\vec{H}']\times[a,A']\cap O=\emptyset.$$

So, in particular, $O\cap((\prod_i H'_i)\times[A']^\omega)$ is clopen in $(\prod_i H'_i)\times[A']^\omega$.

It is easily seen that $\mathbb{PCL}$ is a field of subsets of $\omega^\omega\times[\omega]^\omega$. We will show that $\mathbb{PCL}$ is in fact a σ-field which contains all closed, and therefore all Borel subsets of $\omega^\omega\times[\omega]^\omega$. Before we initiate this job, we present a proposition which explains why it is desirable to prove that the algebra $\mathbb{PCL}$ contains a wide collection of subsets of the product space.

Proposition 5.2. *For every sequence $\{m_i\}_{i=0}^\infty$ of positive integers there is a sequence $\{n_i\}_{i=0}^\infty$ such that for every $\mathbb{PCL}(\vec{m})$-measurable partition*

$$c:(\prod_{i<\omega}n_i)\times[\omega]^\omega\to 2$$

there is $A\in[\omega]^\omega$ and a sequence $\{J_i:i\in\omega\}$ of sets with $J_i\subseteq n_i$ and $|J_i|=m_i$ such that c is constant on $(\prod_{i\in\omega}J_i)\times[A]^\omega$.

Proof. Given $\vec{m}$, consider the associated collection $\mathcal{H} = \mathcal{H}(\vec{m})$. For any $H \in \mathcal{H}$, if c is $\mathbb{PCL}(\vec{m})$-measurable, $c\colon (\prod_{i<\omega} H_i) \times \omega^\omega \to 2$, there is $\vec{H}' \in \mathcal{H}$, $\vec{H}' \leq \vec{H}$ and $A \in [\omega]^\omega$ such that for some expansion level m of $\vec{H}'$, for every $s \in \prod_{i<m} H_i'$,

$$[s, \vec{H}'] \times [A]^\omega \subseteq c^{-1}(0) \text{ or } [s, \vec{H}'] \times [A]^\omega \cap c^{-1}(0) = \emptyset.$$

Thus, $c^{-1}(0)$ is clopen in $\prod_i H_i' \times [A]^\omega$. Consider the partition $d\colon \prod_{i<m} H_i' \to 2$, defined by

$$d(s) = \begin{cases} 0 \text{ iff } [s, \vec{H}'] \times [A]^\omega \subseteq c^{-1}(0), \\ 1 \text{ iff } [s, \vec{H}'] \times [A]^\omega \cap c^{-1}(0) = \emptyset. \end{cases}$$

Since d is a partition into clopen pieces, and the sequence $\{|H_i'| : i \in \omega\}$ dominates the function g of Lemma 2.1, there are sets $K_i \subseteq H_i'$ for all $i < m$, such that $|K_i| = m_i$ for every $i < m$ and d is constant on $\prod_{i<m} K_i$. If we put $K_i = H_i'$ for every $i \geq m$, then, $\vec{K} \in \mathcal{H}(\vec{m})$, $\vec{K} \leq \vec{H}'$ and $\prod_{i\in\omega} K_i \times [A]^\omega$ is contained in $c^{-1}(0)$ or in $c^{-1}(1)$. Notice that since $\vec{H}' \in \mathcal{H}$, $|H_i'| > m_i$ for all i.

It should be clear now that we can take $n_i = |H_i|$ for every i. ∎

We shall need the following well-known concept introduced by Nash-Williams [19] (see also [21], or [23]).

Definition 5.3. Given $A \in [\omega]^\omega$, a family $\mathcal{B}$ of finite subsets of A is a barrier on A if it is an antichain with respect to the partial order given by $\subseteq$ and every infinite subset of A contains an initial segment in $\mathcal{B}$.

For a given family $\mathcal{F}$ of finite subsets of ω, let $\mathcal{F}_a = \{b \in \mathcal{F} : a \sqsubset b\}$, and $\mathcal{F}_a^* = \{b \setminus a : b \in \mathcal{F}_a\}$. If B is an infinite subset of ω, then $\mathcal{F} \restriction B = \mathcal{F} \cap [B]^{<\omega}$. $\hat{\mathcal{F}} = \mathcal{F} \cup \{a : \mathcal{F}_a \neq \emptyset\}$. Notice that if $\mathcal{B}$ is a barrier on A, and $B \in [A]^\omega$, then $\mathcal{B} \restriction B$ is a barrier on B. If $\mathcal{B}$ is a barrier, $(\hat{\mathcal{B}}, \sqsupset)$ is a well-founded tree. The rank of $\mathcal{B}$ is defined as the rank of this tree. For every $n \in \omega$, the only barrier on ω of rank n is $[\omega]^n$.

Proposition 5.4. *If a barrier $\mathcal{B}$ on A has rank α, then, for every $n \in A$, $\mathcal{B}_n^*$ is a barrier on A/n of rank less than α.*

Proof. It is clear that $\mathcal{B}_n^*$ is a barrier on A/n. To see that its rank is less than α, just notice that $\{n\}$ is a node in the tree $(\hat{\mathcal{B}}, \sqsupset)$, and the rank of $\mathcal{B}_n^*$ is just the rank of this node. ∎

We shall also need the following well-known fact shich follows immediately from the Ramsey property of open subsets of $[\omega]^\omega$ (see [7,21,23]).

Lemma 5.5. *Given a family of finite subsets of ω, then there is a $B\in[\omega]^\omega$ such that $\mathcal{F}\restriction B$ is empty or contains a barrier.*

Lemma 5.6. *For every $l\in\omega$, let O_l be an open subset of ω^ω. Given $\vec{H}\in\mathcal{H}$, $n\in\omega$, and $A\in[\omega]^\omega$, there are $\vec{J}\leq_n\vec{H}$ and $B\in[A]^\omega$, and a clopen subset $\mathcal{C}$ of $\prod_{i\in\omega}J$, such that for every $l\in B$, $O_l\cap\prod_{i\in\omega}J_i=\mathcal{C}$.*

Proof. Clearly, we may assume $A=\omega$. By Proposition 3.8 we can assume that for every $k\in\omega$, $O_k\cap\prod_{i\in\omega}H_i$ is clopen, and that there is a sequence $n<n_0<n_1<\dots$ of natural numbers such that for every $s\in\prod_{i<n_k}H_i$, $[s,\vec{H}]\subseteq O_k$ or $[s,\vec{H}]\cap O_k=\emptyset$. We can also assume that n is an expansion level of $\vec{H}$. For every $t\in\bigcup_{n\leq k}\prod_{i<k}H_i$, let $k^t=\max\{i:n_i<|t|\}$. Let $c:\bigcup_{n\leq k}\prod_{i<k}H_i\to 2$ be defined by

$$c(t)=\begin{cases}0 & \text{if } [t\restriction n_{k^t},\vec{H}]\subseteq O_{k^t},\\ 1 & \text{if } [t\restriction n_{k^t},\vec{H}]\cap O_{k^t}=\emptyset.\end{cases}$$

By Proposition 3.6, there are $\vec{J}\leq_n\vec{H}$ and $A'\in[\omega]^\omega$ such that for every $s\in\prod_{i<n}J_i$, c is constant on $[s,\vec{J}]^k$ for every $k\in A'$. Therefore, for every $k\in A'$ and every $s\in\prod_{i<n}J_i$, $[s,\vec{J}]\subseteq O_k$ or $[s,\vec{J}]\cap O_k=\emptyset$. If we list as $\{s_i,s_2,\dots,s_d\}$ the elements of $\prod_{i<n}J_i$, then for every $k\in A'$,

$O_k\cap\prod_{i\in\omega}J_i$ is of the form $\cup\{[s_i,\vec{J}]:i\in F\}$ for some $F\subseteq\{1,\dots,d\}$. Thus, we can find $B\in[A']^\omega$ and a single such F such that for every $k\in B$, $O_k\cap\prod_{i\in\omega}J_i=\cup\{[s_i,\vec{J}]:i\in F\}=\mathcal{C}$. ∎

Lemma 5.7. *Let $\mathcal{B}$ be a barrier, and for every $b\in\mathcal{B}$, let $O_b\subseteq\omega^\omega$ be an open set. Given $\vec{H}\in\mathcal{H}$, $A\in[\omega]^\omega$ and $n\in\omega$, there are $\vec{K}\leq_n\vec{H}$, $B\in[A]^\omega$ and a clopen subset $\mathcal{C}$ of $\prod_{i\in\omega}K_i$ such that for every $b\in\mathcal{B}\restriction B$, $O_b\cap\prod_{i\in\omega}K_i=\mathcal{C}$.*

Proof. We prove the lemma by induction on the rank of the barrier $\mathcal{B}$. The case of rank 1 is covered by Lemma 5.6. Suppose the rank of $\mathcal{B}$ is α and that for every barrier of rank $<\alpha$, we have the result.

There are barriers $\mathcal{B}_i$ for every $i\in\omega$ of rank $<\alpha$ such that every $b\in\mathcal{B}$ is of the form $\{n\}\cup b'$ for some $b'\in\mathcal{B}_n$. In other words, $\mathcal{B}=\{\{n\}\cup b:b\in\mathcal{B}_n\}$.

For every $n\in\omega$ and $b\in\mathcal{B}_n$, let $O_{n,b}=O_{\{n\}\cup b}$.

By inductive hypothesis, for every $n\in\omega$, there are $\vec{H}^0\leq_n\vec{H}$, $A^0\in[A]^\omega$, and a clopen set $\mathcal{C}_0\subseteq\prod_{i\in\omega}H^0_i$ such that for every $b\in\mathcal{B}_0\restriction A^0$, $O_{0,b}\cap\prod_{i\in\omega}H^0_i=\mathcal{C}_0$.

Since C_0 is clopen, there is a level l_0 of $\vec{H}^0$ at which membership in it is decided. Let n_0 be the second expansion level of $\vec{H}^0$ above l_0, and apply the inductive hypothesis again to $\vec{H}^0$, A^0,n_0 to obtain $\vec{H}^1\leq_{n_0}\vec{H}^0$,

$A^1 \in [a^0]^\omega$ and a clopen subset $\mathcal{C}_1 \subseteq \prod_{i\in\omega} H_i^1$ such that for every $b \in \mathcal{B}_1 \restriction A^1$, $O_{1,b} \cap \prod_{i\in\omega} H_i^1 = \mathcal{C}_1$. We can assume that $a_0 = \min(A^0) < a_1 = \min(A^1)$. Having defined $\vec{H} \geq_n \vec{H}^0 \geq_{n_0} \vec{H}^1 \geq_{n_1} \cdots \geq_{n_{k-1}} \vec{H}^k$, and $A \supseteq A^0 \supseteq \cdots \supseteq A^k$ with $a_l = \min(A^l) < a_{l+1} = \min(a^{l+1})$ for every $l < k$, and clopen $\mathcal{C}_l \subseteq \prod_{i\in\omega} H_i^l$ such that for every $l \leq k$ and every $b \in \mathcal{B}_l \restriction A^l$ $O_{l,b} \cap \prod_{i\in\omega} H_i^l = \mathcal{C}_l$. Let l_k be such that membership in $\mathcal{C}_k$ is decided at level l_k of $\vec{H}^k$. Let n_{k+1} be the second expansion level of $\vec{H}^k$ above l_k. By the inductive hypothesis, there are $\vec{H}^{k+1} \leq_{n_{k+1}} \vec{H}^k$, $A^{k+1} \in [A^k]^\omega$ and $\mathcal{C}_{k+1}$ clopen subset of $\prod_{i\in\omega} H_i^{k+1}$, such that for every $b \in \mathcal{B}_{k+1} \restriction A^{k+1}$, $O_{k+1,b} \cap \prod_{i\in\omega} H_i^{k+1} = \mathcal{C}_{k+1}$. We can assume that $a_{k+1} = \min(A^{k+1}) > a_k$.

Let $\vec{J}$ be the fusion of the $\vec{H}^n$ and $D = \{a_0, a_1, \dots\}$, for every $n \in \omega$, if $b \in \mathcal{B}_n \restriction D$, $O_{n,b} \cap \prod_{i\in\omega} J_i = \mathcal{C}_n$. Apply now the Lemma 5.6 to $\vec{J}$ and $\mathcal{C}_n$ to obtain $\vec{K} \leq_n \vec{H}$, $B \in [A]^\omega$ and $\mathcal{C}$ as desired. ∎

Now we come to the main lemma of this section.

Lemma 5.8. *Every open subset of $\omega^\omega \times [\omega]^\omega$ belongs to $\mathbb{PCL}$.*

Proof. Let O be a given open subset of $\omega^\omega \times [\omega]^\omega$.

Let $\vec{H} \in \mathcal{H}(\vec{m})$, $A \in [\omega]^\omega$, $n \in \omega$ and $a \in [A]^{<\omega}$, be given inputs to test whether $\mathcal{O}$ belongs to $\mathbb{PCL}$. We need to find $\vec{J} \leq_n \vec{H}$, $B \in [A]^\omega$ and $l \geq n$ such that for every $s \in \prod_{i<l} J_i$ either $[s, \vec{J}] \times [a, B] \subseteq \mathcal{O}$ or $[s, \vec{J}] \times [a, B] \cap \mathcal{O} = \emptyset$. By Lemma 4.4 we find $\vec{H}^1 \leq_n \vec{H}$, $A^1 \in [A]^\omega$, and $M \in [\omega]^\omega$ such that $n = \min M$ and such that $(\vec{H}^1 A^1)$ decides $(s, a \cup b)$ for all $s \in \bigcup_{k\in M} \prod_{i<k} H_i^1$ and $b \subseteq A^1$ such that $\max(b) < |s|$. We may assume $\min(A^1) > \max(a)$. For $b \in [A^1]^\omega$, let

$$\mathcal{O}_b = \cup\{[t, \vec{H}^1] : t \in \bigcup_k \prod_{i<k} H_i^1 \text{ and } [t, \vec{H}^1] \times [a \cup b, A^1] \subseteq \mathcal{O}\}.$$

Let

$$\mathcal{F} = \{b \in [A^1]^\omega : b \neq \emptyset \text{ and } \mathcal{O}_b \neq \emptyset\}.$$

By Lemma 5.5 there is $B \in [A]^\omega$ such that either:

Case 1. $\mathcal{F} \cap [B]^\omega = \emptyset$.

We claim that in this case $(\prod_{i\in\omega} H_i^1) \times [a, B] \cap \mathcal{O} = \emptyset$. Otherwise, there exist $x \in \prod_{i\in\omega} H_i^1$ and $X \in [a, B]$ such that $(x, X) \in \mathcal{O}$. Since $\mathcal{O}$ is open, there exist k and l such that $b = X \cap \{0, \dots, l\} \setminus a \neq \emptyset$, and

$$[x \restriction k] \times [X \cap \{0, \dots, l\}, \omega] \subseteq \mathcal{O}.$$

Then $\emptyset \neq [x \restriction k, \vec{H}^1] \subseteq \mathcal{O}_b$. It follows that $b \in \mathcal{F} \cap [B]^\omega$, a contradiction.

Case 2. $\mathcal{F}\cap[B]^\omega$ contains a barrier $\mathcal{F}_0$ on B. By Lemma 5.7 there exist an infinite $C\subseteq B$, $\vec{J}\leq_n \vec{H}^1$ in $\mathcal{H}$, and a relatively clopen set $\mathcal{C}\subseteq\prod_{i\in\omega}J_i$ such that

$$\mathcal{O}_b\cap\prod_{i\in\omega}J_i=\mathcal{C}$$

for all $b\in\mathcal{F}_0\restriction C$.

Pick $\bar{n}>n$ in M such that $\mathcal{C}\cap\prod_{i\in\omega}J_i$ depends only on coordinates $<\bar{n}$.

Claim. Suppose $s\in\prod_{i<\bar{n}}J_i$ is such that $[s,\vec{J}]\cap\mathcal{C}=\emptyset$. Then, for every $D\subseteq C$ and $b\in\mathcal{F}_0\restriction C$ there exist an infinite set $E\subseteq D$ such that $([s,\vec{J}]\times[a\cup b,E])\cap\mathcal{O}=\emptyset$.

Proof of Claim. Let $\mathcal{G}_b=\{\emptyset\neq c\in[C/b]^{<\omega}:\mathcal{O}_{b\cup c}\cap[s,\vec{J}]\neq\emptyset\}$. By Lemma 5.5 there is an infinite $E\subseteq D/b$ such that one of the two subcases holds.

Case 2.1. $[E]^{<\omega}\cap\mathcal{G}_b=\emptyset$. As in Case 1, one can show that in this subcase we have that

$$[s,\vec{J}]\times[a\cup b,E]\cap\mathcal{O}=\emptyset$$

as required.

Case 2.2. $[E]^{<\omega}\cap\mathcal{G}_b=\emptyset$ contains a barrier $\mathcal{G}_b^*$ on E. Applying Lemma 5.7, we find an infinite $F\subseteq E$, $\vec{K}\leq_{\bar{n}}\vec{J}$, and a nonempty relatively clopen set $\mathcal{D}\subseteq[s,\vec{K}]$ such that $\mathcal{O}_{b\cup c}\cap[s,\vec{K}]=\mathcal{D}$ for all $c\in\mathcal{G}_b^*\restriction F$.

Choose $l\in M/b$ so that $\mathcal{D}$ depends only on coordinates $<l$. Pick an $\bar{s}\in\prod_{i<l}K_i$ extending s such that $[s,\vec{K}]\subseteq\mathcal{D}$. Going back to the definition of $\mathcal{O}_{b\cup c}$, we infer that $[\bar{s},\vec{H}^1]\subseteq\mathcal{O}_{b\cup c}$ for all $c\in\mathcal{G}_b^*\restriction F$. Since $\mathcal{G}_b^*\restriction F$ is a barrier on F, going back to the definition of $\mathcal{O}_{b\cup c}$ we conclude that

$$[\bar{s},\vec{H}^1]\times[a\cup b,F]\subseteq\mathcal{O}.$$

By the choice of $\vec{H}^1, A^1$ and M, we know that $(\vec{H}^1,A^!)$ decides $(\bar{s},b)$. Since (H^1,F) accepts $(\bar{s},b)$ by Lemma 4.2(1), we conclude that $(\vec{H}^1,A^1)$ accepts $(\bar{s},b)$. It follows that

$$[\bar{s},\vec{H}^1]\times[a\cup b,F]\subseteq\mathcal{O},$$

and therefore $[\bar{s},\vec{H}^1]\subseteq\mathcal{O}_b$. Since $\bar{s}$ extends s, which was chosen to avoid the clopen subset $\mathcal{C}$ of $\prod_{i\in\omega}J_i$, we conclude that

$$\mathcal{O}_b\cap\Big(\prod_{i\in\omega}J_i\Big)\neq\mathcal{C}.$$

a contradiction. This shows that Case 2.2 is impossible, finishing the proof of the Claim. ∎

Using the Claim we build a decreasing sequence

$$C = C_0 \supseteq C_1 \supseteq \ldots \supseteq C_k \supseteq \ldots$$

of infinite subsets of C such that the sequence $\{n_k = \min(C_k)\}_{k\in\omega}$ is a strictly increasing sequence of integers such that for every k, every $b \subseteq \{n_0,\ldots,n_k\}$, $b\in\mathcal{F}_0$, and every $s\in\prod_{i<\bar{n}} J_i$ with $[s,\vec{J}]\cap\mathcal{C}=\emptyset$, we have that

$$([s,\vec{J}] \times [a\cup b, C_{k+1}]) \cap \mathcal{O} = \emptyset.$$

Let $C_\infty = \{n_k : k\in\omega\}$. Then, we have obtained $C_\infty \subseteq A$, $\bar{n}\geq n$, and $\vec{J}\leq_n \vec{H}$ in $\mathcal{H}$ satisfying the conclusion of the statement of the lemma, i.e., for all $s\in\prod_{i<\bar{n}} J_i$, either

$$[s,\vec{J}] \times [a, C_\infty] \subseteq \mathcal{O}, \text{ or } [s,\vec{J}] \times [a, C_\infty] \cap \mathcal{O} = \emptyset.$$

This finishes the proof. ∎

Lemma 5.9. *The field $\mathbb{PCL}$ is closed under countable unions.*

Proof. Suppose $B=\bigcup_{j\in\omega} B_j$ and that each set B_j is in $\mathbb{PCL}$.

Let $\vec{H}\in\mathcal{H}$, $A\in[\omega]^\omega$ and $n_0\in\omega$ be given, and let $a=\{a_0,\ldots,a_{m-1}\}\subseteq A$. Since $B_0\in\mathbb{PCL}$, there is a level $l\geq n_0$ of $\vec{H}$, $\vec{H}^0\leq_l \vec{H}$ and $A^0\in[a,A]^\omega$ such that for every $s\in\prod_{i<l} H_i^0$, $[s,\vec{H}^0]\times[a,A^0]\subseteq B_0$ or $[s,\vec{H}^0]\times[a,A^0]\cap B_0=\emptyset$. Let $a_m\in A^0$ be the first element of A^0 above a_{m-1} and pick n_1 above the first expansion level of $\vec{H}^0$ which is above n_0.

Assume we have defined $\vec{H}^j$, A^j, a_{m+j} and n_j for every $j<k$.

Let n_k be above the first expansion level of $\vec{H}^{k-1}$ which is greater than n_{k-1}. Since B_k is in $\mathbb{PCL}$, there is $\vec{H}^k\leq_{n_k} H^{k-1}$, and $A^k\subseteq A^{k-1}$ with $\{a_0,\ldots,a_{m+k-1}\}\subseteq A^k$, such that for every $s\in\prod_{i<n_k} H_i^k$, $[s,\vec{H}^k]\times[a,A']\subseteq B_k$ or $[s,\vec{H}^k]\times[a,A']\cap B_k=\emptyset$. Take a_{m+k} the first element of A^k above a_{m+k-1} and n_{k+1} above the first expansion level of $\vec{H}^k$ above n_k. This way we complete the inductive definition. Put $A'=\{a_0,a_1,\ldots,a_k,\ldots\}$, and let $\vec{H}'$ be the fusion of the sequence $\vec{H}^0\geq_{n_0}\vec{H}^1\geq_{n_1}\ldots\vec{H}^k\geq_{n_k}\ldots$.

Each B_j is clopen in $(\prod_i H_i')\times[A']^\omega$, therefore B is open in this product. Applying the previous lemma, we get $\vec{H}''\leq_{n_0}\vec{H}'$ and $A''\in[A']^\omega$ such that for every $s\in\prod_{i<n} H_i''$ for some expansion level $n\geq n_0$, $[s,\vec{H}'']\times[a,A'']\subseteq B$ or $[s,\vec{H}'']\times[a,A'']\cap B=\emptyset$. ∎

Corollary 5.10. *All Borel subsets of $\omega^\omega\times[\omega]^\omega$ belong to the algebra $\mathbb{PCL}$.*

Proof. This follows from Lemmas 5.8 and 5.9. ∎

As a result, using Proposition 5.2 and working under the assumption of Lemma 2.2, we get the following.

Theorem 5.11. *For every sequence $\vec{m}=\{m_i\}_{i=0}^{\infty}$ of positive integers there is a sequence $\vec{n}=\{n_i\}_{i=0}^{\infty}$ of positive integers such that for every Borel map $c\colon(\prod_{i=0}^{\infty} n_i)\times[\omega]^{\omega}\to 2$, there exist $H\in[\omega]^{\omega}$ and $H_i\subseteq n_i$, $|H_i|=m_i$ (for all $i\in\omega$) such that c is constant on $(\prod_{i=0}^{\infty} H_i)\times[H]^{\omega}$.*

6. A parametrized Borel–Ramsey theorem

We are now ready to state and prove the result claimed in the abstract of the paper.

Theorem 6.1. *For every Borel coloring $c\colon\omega^{\omega}\times[\omega]^{\omega}\to 2$ and every sequence $\{m_i\}_{i=0}^{\infty}$ of positive integers, there exist $H\in[\omega]^{\omega}$ and a sequence $\{H_i\}_{i=0}^{\infty}$ of subsets of ω with $|H_i|=m_i$ for all i, such that c is constant on the product $(\prod_{i\in\omega} H_i)\times[H]^{\omega}$.*

Proof. By Theorem 5.11 this follows from the assumption of Lemma 2.2 under which we have worked through the sections 2–5. Thus, in particular, the conclusion of the theorem follows assuming that all $\boldsymbol{\Sigma}^1_2$ subsets of $[\omega]^{\omega}$ are Ramsey. So, in particular, Theorem 6.1 holds in any forcing extension of the universe satisfying Martin's axiom and the negation of the Continuum Hypothesis. Since the conclusion of the theorem is a $\boldsymbol{\Sigma}^1_2$ sentence in the parameters $\{m_i\}_{i=0}^{\infty}$ and c, we finish using Shoenfield's absoluteness theorem (see [13, 22]). ∎

So far, we have considered partitions into 2 rather than into any finite number of pieces, but this is not an essential restriction. This is clear for the result of Theorem 5.11 by a repeated application of the same result.

More informative results can be given regarding how much the sequence $\vec{n}$ has to increase, given a sequence $\vec{m}$, when we pass from partitions into k pieces to partitions into $k+1$ pieces. For example, one can prove the following.

Proposition 6.2. *Suppose $\mathbb{B}$ is a σ-field of subsets of $\omega^{\omega}\times[\omega]^{\omega}$ containing all Borel sets and closed under continuous substitutions.. Given a sequence $\vec{m}=\{m_i\}_{i=0}^{\infty}$ of positive integers and a sequence $\vec{n}=\{n_i\}_{i=0}^{\infty}$ such that for every $\mathbb{B}$-measurable partition*

$$c:\Big(\prod_{i<\omega} n_i\Big)\times[\omega]^{\omega}\to k$$

there is $A\in[\omega]^\omega$ and a sequence $\{H_i : i\in\omega\}$ of sets with $H_i\subseteq n_i$ and $|H_i|=m_i$ such that c is constant on $(\prod_{i\in\omega} H_i)\times[A]^\omega$, then the sequence $\{n_{2i+1}\}_{i=0}^\infty$ satisfies that for every $\mathbb{B}$-measurable partition

$$c:\Big(\prod_{i<\omega} n_{2i+1}\Big)\times[\omega]^\omega\to k+1$$

there is $A\in[\omega]^\omega$ and a sequence $\{H_i : i\in\omega\}$ of sets with $H_i\subseteq n_{2i+1}$ and $|H_i|=m_i$ such that c is constant on $(\prod_{i\in\omega} H_i)\times[A]^\omega$.

Proof. Suppose $\vec{n}=\{n_i\}_{i=0}^\infty$ and $\vec{m}=\{m_i\}_{i=0}^\infty$ are increasing sequences such that for every $\mathbb{B}$-measurable partition $c:(\prod_{i<\omega} n_i)\times[\omega]^\omega\to k$ there is $H\in[\omega]^\omega$ and a sequence $\{H_i : i\in\omega\}$ of sets with $H_i\subseteq n_i$ and $|H_i|=m_i$ such that c is constant on $(\prod_{i\in\omega} H_i)\times[H]^\omega$.

We will adapt an argument from [3] to show that the sequence $\vec{n}'=\{n_i'\}_{i=0}^\infty$ defined by $n_i'=n_{2i+1}$ for every $i\in\omega$ satisfies the partition relation for $\mathbb{B}$-measurable partition into $k+1$ pieces. For a sequence $\vec{x}=\{x_i\}_{i=0}^\infty$, let $\vec{x}_e=\{x_{2i}\}_{i=0}^\infty$, and $\vec{x}_o=\{x_{2i+1}\}_{i=0}^\infty$.

Given a $\mathbb{B}$-measurable function $f:(\prod_{i<\omega} n_i')\times[\omega]^\omega\to k+1$ define $D(x,A)=f(x_o,A)-f(x_e,A)$. By our assumption on $\mathbb{B}$, the map D is $\mathbb{B}$-measurable. Using D, define an auxiliary $\mathbb{B}$-measurable partition $g:\prod_{i=0}^\infty n_i\times[\omega]^\omega\to k$ as follows:

$$\begin{aligned} g(x,A) &= 0, \text{if } D(x,A)\in\{0,-2\};\\ &= 1, \text{if } D(x,A)\in\{-1,1,2\};\\ &= 2, \text{if } D(x,A)\in\{-3,3\};\\ &\ \ \vdots\\ &= k-1, \text{if } D(x,A)\in\{-k,k\}.\end{aligned}$$

By hypothesis, there is a sequence $\{H_i\}_{i=0}^\infty$ and a set $H\in[\omega]^\omega$ such that $|H_i|=m_i$ for all i and g is constant on $\prod_{i<\omega} H_i\times[H]^\omega$. We claim that either $\vec{H}_e$ or $\vec{H}_o$ works for f. Otherwise, f takes at least two different values $\{a,b\}$ on $\prod_{i=0}^\infty H_{2i}\times[H]^\omega$ and at least two different values $\{c,d\}$ on $\prod_{i=0}^\infty H_{2i+1}\times[H]^\omega$, and analyzing how $\{a,b\}$ and $\{c,d\}$ can be related to each other we reach a contradiction. Consider the differences $c-a$, $c-b$, $d-a$ and $d-b$. Clearly, $c–a\neq c–b$, $c–a\neq d–a$, $c–b\neq d–b$ and $d–a\neq d–b$. The four differences cannot be all distinct, since by the definition of g and by homogeneity of $(\vec{H},H)$, the number of different values among them can only be two or three. Now, if $c-b=d-a$, then $c-a\neq d-b$, since otherwise we would get $c=d$ (thus if $c-a=d-b$, then $c-b\neq d-a$); therefore D takes three different values on

$\prod_{i=0}^{\infty} H_i \times [H]^{\omega}$, and it follows that g must take the constant value 1 on this product.

If $c-b=d-a$, then this value cannot be 1, since this means that $c=b+1$ and $-a=1-d$, and then $c-a=b+1+1-d$, thus $(c-a)+(d-b)=2$. This is impossible since the other possible values for $c-a$ and $d-b$ are -1 and 2, and their sum is not 2. Reasoning the same way we see that $c-b=d-a$ cannot be -1 nor 2.

Similarly, if $c-a=d-b$, this value cannot be 1, -1 nor 2.

Now, since f is constant on at least one of the products $(\prod_{i=0}^{\infty} H_{2i}) \times [H]^{\omega}$ or $(\prod_{i=0}^{\infty} H_{2i+1}) \times [H]^{\omega}$, and since $H_i \subseteq n_i \leq n_i'$ for all i, and $m_i \leq |H_{2i}| \leq |H_{2i+1}|$, we have the desired result. ∎

It would be interesting to give explicit sequences satisfying parametrized versions of polarized partitions. For example, in [4] we have shown that if A is the Ackermann function, the partition relation

$$\begin{pmatrix} A(3) \\ A(6) \\ A(9) \\ \vdots \end{pmatrix} \to \begin{pmatrix} 2 \\ 2 \\ 2 \\ \vdots \end{pmatrix}$$

holds in the realm of Borel partitions. Is it possible to parametrize this partition relation by $\omega \to (\omega)^{\omega}$?. For this, it seems necessary to have an explicitly defined function having the properties of function h of Lemma 2.2.

We conclude with some comments about extensions of the results presented here. It can be shown that for each sequence $\vec{m}$, the algebra $\mathbb{PCL}(\vec{m})$ is closed under Suslin's operation and therefore it contains the analytic subsets of the product space $\omega^{\omega} \times [\omega]^{\omega}$. The proof uses methods which are outside the scope of this paper and will appear elsewhere.

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